THE MIXED OSCILLATION PROBLEM FOR AN INFINITE PLATE OF UNIT WIDTH

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The periodic oscillations of an infinite plate have been studied. The problem, when solved by the method of [1], reduces to the Riemann boundary problem [2]. The latter problem is solved approximately. The substance of the approximation consists in the replacement of coefficients having a comparatively complex structure, thus permitting the coefficients to be expressed in a more simple form. Then, by an inverse Fourier transformation applied to the solution of the Riemann problem, an approximate solution to the original problem is found. The solution is carried to the point of numerical results and the error in the approximation is estimated.

1. We consider the equation

$$\Delta\Delta u - \mu^4 u = 0 \qquad (u = u(x, y), \mu \text{ — numerical parameter}) \tag{1.1}$$

on the strip $0 < y < 1$, $-\infty < x < \infty$, with the following boundary conditions:

$$\begin{gathered} u(x, 1) = 0, \qquad u_{yy}(x, 1) = 0, \qquad u(x, 0) = 0, \qquad -\infty < x < \infty \\ u_y(x, 0) = 0, \quad x > 0, \quad u_{yy}(x, 0) = f(x), \qquad x < 0 \end{gathered} \tag{1.2}$$

Equation (1.1) determines the amplitude of periodic oscillations of a thin plate, and conditions (1.2) indicate that the upper edge of the plate ($y = 1$) is hinged, that the right-hand half of the lower edge ($x > 0$, $y = 0$) is fixed and that the left-hand half of the lower edge ($x < 0$, $y = 0$) has a bending moment.

A Fourier transformation will be used, and so for conditions (1.2) the derivatives $u_y(x, 0)$ and $u_{yy}(x, 0)$ are defined for all real axes.

We introduce functions $\varphi_+(x) \equiv 0$, $\varphi_-(x) \equiv 0$ for $x < 0$ and $x > 0$, by means of which $f(x)$ is defined as identically zero for $x > 0$; conditions (1.2) are written in the form

$$u(x, 1) = 0, \qquad u_{yy}(x, 1) = 0, \qquad u(x, 0) = 0$$
$$u_y(x, 0) = \varphi_-(x), \qquad u_{yy}(x, 0) = f_-(x) + \varphi_+(x) \qquad (-\infty < x < \infty) \qquad (1.3)$$

The object of this work is to determine the functions $\varphi_+(x) = u_{yy}(x, 0)$ and $\varphi_-(x) = u_y(x, 0)$ for $x > 0$ and $x < 0$, respectively.

If a Fourier transformation in x is applied to equation (1.1) and to the boundary conditions (1.2), then equation (1.1) leads to an ordinary differential equation

$$\frac{d^4U}{dy^4} - 2x^2\frac{d^2U}{dy^2} + (x^4 - \mu^4)U = 0 \qquad (1.4)$$

in which the variable x is a parameter, and the boundary conditions (1.3) give

$$U(x, 1) = 0, \qquad U_{yy}(x, 1) = 0, \qquad U(x, 0) = 0$$
$$U_y(x, 0) = \Phi^-(x), \qquad U_{yy}(x, 0) = F^-(x) + \Phi^+(x) \qquad (-\infty < x < \infty) \qquad (1.5)$$

where $\Phi^+(x)$, $\Phi^-(x)$ and $F^-(x)$ will be limiting values of functions analytically corresponding to the upper and lower semiplanes [3].

The general solution of equation (1.4) has the form

$$U(x, y) = A(x)e^{y\alpha_1} + B(x)e^{-y\alpha_1} + C(x)e^{y\alpha_2} + D(x)e^{-y\alpha_2}$$
$$(\alpha_1 = \sqrt{x^2 + \mu^2}, \quad \alpha_2 = \sqrt{x^2 - \mu^2})$$

Here A, B, C and D are arbitrary functions of x.

Substitution of the function $U(x, y)$ and its derivatives U_y and U_{yy} into conditions (1.5) gives five relations among the functions A, B, C, D, Φ^+ and Φ^-, from which by exclusion of A, B, C and D we obtain the Riemann boundary problem; the functions $\Phi^+(z)$ and $\Phi^-(z)$ analytically corresponding to the upper and lower semiplanes are found, satisfying this relation on the real axis

$$\Phi^+(x) = G(x)\Phi^-(x) - F^-(x) \qquad (-\infty < x < \infty) \qquad (1.6)$$

where

$$G(x) = \frac{-2\mu^2}{\sqrt{x^2 + \mu^2}\coth\sqrt{x^2 + \mu^2} - \sqrt{x^2 - \mu^2}\coth\sqrt{x^2 - \mu^2}} \qquad (1.7)$$

2. The Riemann problem with boundary conditions (1.6) is solved approximately. We note that an exact solution may be found if $\Phi^+(x)$ and

$\Phi^-(x)$ are expressed by integrals of the Cauchy type. Nevertheless, such a solution is not obtained here since the functions $\varphi_+(x) = u_{yy}(x, 0)$ and $\varphi_-(x) = u_y(x, 0)$ are found by an inverse Fourier transformation from the solution of the Riemann problem, while the calculation when Φ^+ and Φ^- are expressed by Cauchy integrals is difficult. The following example is presented to show this.

Following Koiter [4], we substitute a rational function for the coefficient (1.7) in the boundary condition (1.6). Then the solution of the approximate Riemann problem will have a simple expression from which the inverse Fourier transformation can be easily calculated.

In order that the error arising from the substitution of an approximate solution for the exact one not be too large, it is necessary to clarify in what sense the coefficient $G(x)$ is a solution of the Riemann problem. The following problem is solved in relation to the class of solutions. For example, if in the original problem the function $\varphi_-(x)$ is considered to be in the class $L_2(-\infty, 0)$ with the usual norm, then, as is made clear in Section 4, in determining the error one must strive for the smallest value of the maximum modulus of the difference between the exact and approximate coefficients of the Riemann problem (see formula (4.4)). This has been indicated by Cherskii [5].

We return to relations (1.6). By substitution of the approximate coefficient $G(x)$ we find the following properties of the function (1.7): it is even, it has no real roots, and it holds to infinity. $G(x)$ is presented in the form of a product

$$G(x) = -2\mu^2\sqrt{x^2+\mu^2}\,G_1(x) \tag{2.1}$$

and for $G_1(x)$ we substitute the rational fraction

$$P(x) = \sum_{k=0}^{n} \frac{a_k x^{2k}}{b_k x^{2k}}$$

For the problem under consideration it is sufficient to take $n = 2$, for which we set $a_2 = b_2 = 1$. After substitution we obtain a Riemann problem with the boundary condition

$$\Psi^+(x) = -2\mu^2\sqrt{x^2+\mu^2}\,\frac{x^4+a_1x^2+a_0}{x^4+b_1x^2+b_0}\,\Psi^-(x) - F^-(x) \qquad (-\infty < x < x) \tag{2.2}$$

The coefficients a_0, a_1, b_0 and b_1 and determined from a linear system obtained by equating $G_1(x)$ and $P(x)$ at four values of x. For example, if μ is set equal to 1 in equation (1.1) and values of x are fixed as 0, 1, 2 and 5, we obtain

$$a_0 = 9.3745, \quad a_1 = 4.3305, \quad b_0 = 6.2898, \quad b_1 = 5.0195 \tag{2.3}$$

If in $P(x)$ the numerators of the roots are denoted by $x_{1,2} = \pm(\alpha + i\beta)$ and $x_{3,4} = \pm(\alpha - i\beta)$, and the denominators by $x_{5,6} = \pm(\gamma + i\delta)$ and $x_{7,8} = \pm(\gamma - i\delta)$, where α, β, γ and δ are positive numbers, the $P(x)$ is in the form of a ratio

$$P(x) = \frac{X^+(x)}{X^-(x)}, \qquad X^+(x) = \frac{(x - \alpha + i\beta)(x + \alpha + i\beta)}{(x - \gamma + i\delta)(x + \gamma + i\delta)}$$
$$X^-(x) = \frac{(x + \gamma - i\delta)(x - \gamma - i\delta)}{(x + \alpha - i\beta)(x - \alpha - i\beta)} \tag{2.4}$$

Let $\sqrt{(x^2 + \mu^2)} = -\sqrt{(x + i|\mu|)}\sqrt{(x - i|\mu|)}$. We choose a positive value of the root $\sqrt{(x^2 + \mu^2)}$. For $\sqrt{(x + i|\mu|)}$ we take a branch determined by the equality of $\sqrt{(x + i|\mu|)}_{x=+0} = \sqrt{(\mu/2)}(1 + i)$; then the branch of the root $\sqrt{(x - i|\mu|)}$ is automatically determined, and the relation (2.2) may be written in the form

$$\Psi^+(x) = \frac{2\mu^2 \sqrt{x + i|\mu|}\sqrt{x - i|\mu|}\, X^+(x)}{X^-(x)} \Psi^-(x) - F^-(x)$$

From this it follows that

$$\frac{\Psi^+(x)}{\sqrt{x + i|\mu|}\, X^+(x)} = \frac{2\mu^2 \sqrt{x - i|\mu|}\, \Psi^-(x)}{X^-(x)} - \frac{F^-(x)}{\sqrt{x + i|\mu|}\, X^+(x)} \tag{2.5}$$

The free member in this equality may be represented by the Sokhotskii formula

$$\frac{F^-(x)}{\sqrt{x + i|\mu|}\, X^+(x)} = \left[\frac{F^-(x)}{\sqrt{x + i|\mu|}\, X^+(x)}\right]^+ - \left[\frac{F^-(x)}{\sqrt{x + i|\mu|}\, X^+(x)}\right]^- \tag{2.6}$$

where the plus and minus signs on the brackets signify that the right-hand sums are limiting values of a Cauchy type integral with a density $F^-(x)[\sqrt{(x + i|\mu|)}X^+(x)]^{-1}$ when $z \to x$ corresponding to the upper and lower semiplanes.

It follows from (2.5) and (2.6) that

$$\frac{\Psi^+(x)}{\sqrt{x + i|\mu|}\, X^+(x)} + \left[\frac{F^-(x)}{\sqrt{x + i|\mu|}\, X^+(x)}\right]^+ =$$
$$= \frac{2\mu^2 \sqrt{x - i|\mu|}\, \Psi^-(x)}{X^-(x)} + \left[\frac{F^-(x)}{\sqrt{x + i|\mu|}\, X^+(x)}\right]^-$$

By an application of the principle of analytic continuation and the generalized Liouville theorem to this equality we get for the solution to the approximate Riemann problem

$$\Psi^+(x) = -\sqrt{x + i|\mu|}\, X^+(x)\left[\frac{F^-(x)}{\sqrt{x + i|\mu|}\, X^+(x)}\right]^+ \tag{2.7}$$

$$\Psi^-(x) = -\frac{X^-(x)}{2\mu^2\sqrt{x - i|\mu|}}\left[\frac{F^-(x)}{\sqrt{x + i|\mu|}\,X^+(x)}\right]^- \qquad (2.7)$$

3. Approximate expressions for the functions $\varphi_+(x) = u_{yy}(x, 0)$ and $\varphi_-(x) = u_y(x, 0)$ are found by inverse Fourier transformations of the functions $\Psi^+(x)$ and $\Psi^-(x)$. We limit ourselves to a determination of $\Psi_-(x)$: the approximation for $\varphi_-(x) = u_y(x, 0)$.

For the calculations we use the relations

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-ixt}\,dt}{t+a} = \begin{cases} -i\sqrt{2\pi}\,\eta(x)\,e^{iax} & (\operatorname{Im} a > 0) \\ i\sqrt{2\pi}\,\eta(-x)\,e^{iax} & (\operatorname{Im} a < 0)\end{cases} \qquad (3.1)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-ixt}\,dt}{\sqrt{t+ib}} = \begin{cases} (1-i)\,\eta(x)\,e^{-bx}/|\sqrt{x}| & (\operatorname{Re} b \geqslant 0) \\ -(1-i)\,\eta(-x)\,e^{-bx}/|\sqrt{x}| & (\operatorname{Re} b \leqslant 0)\end{cases} \qquad (3.2)$$

$$\eta(t) = 1 \quad \text{for } t > 0, \qquad \eta(t) = 0 \quad \text{for } t < 0)$$

We expand $[X^+(x)]^{-1}$ and $X^-(x)$ as simple fractions. We obtain, taking (2.4) into account

$$[X^+(x)]^{-1} = 1 + \frac{S}{x - \alpha + i\beta} + \frac{T}{x + \alpha + i\beta}$$

$$X^-(x) = 1 + \frac{M}{x - \alpha - i\beta} + \frac{N}{x + \alpha - i\beta}$$

The formulas obtained are substituted into the expression for $\Psi^-(x)$; application of the properties of the Fourier transformation, as well as relations (3.1) and (3.2), gives for the function $\psi_-(x)$

$$\psi_-(x) = \frac{1}{2\pi\mu^2}\int_x^0 \frac{e^{|\mu|(x-t)}}{|\sqrt{x-t}|}\left[w(t) + \sum_{k=1}^{2} p_k v_k(t)\right]dt +$$

$$+\frac{1}{2\pi\mu^2}\int_x^0 \sum_{k=3}^{8}\frac{p_k}{\sqrt{q_k}}\, e^{r_k(x-t)} F\left(\sqrt{q_k|x-t|}\right) v_k(t)\,dt \qquad (x < 0)$$

In this we have

$$F(z) = \int_0^z e^{x^2}\,dx, \quad w(t) = -\int_{-\infty}^{t}\frac{e^{-|\mu|(t-u)}}{|\sqrt{t-u}|} f_-(u)\,du \qquad (t < 0)$$

$$v_k(t) = \frac{1}{\sqrt{m_k}}\int_{-\infty}^{t} e^{n_k(t-u)} F\left(\sqrt{m_k|t-u|}\right) f_-(u)\,du \qquad (t < 0) \qquad (k = 1, 2, 5, 6, 7, 8)$$

$$v_k(t) = w(t) \qquad (k = 3, 4)$$

$$\begin{array}{lll} p_1 = 2iS, & p_5 = -4MS, & m_1 = m_5 = m_7 = q_3 = q_5 = q_6 = \beta - |\mu| + i\alpha \\ p_2 = 2iT, & p_6 = -4MT, & m_2 = m_6 = m_8 = q_4 = q_7 = q_8 = \beta - |\mu| - i\alpha \\ p_3 = 2iM, & p_7 = -4NS, & -n_1 = -n_5 = -n_7 = r_4 = r_7 = r_8 = \beta + i\alpha \\ p_4 = 2iN, & p_8 = -4NT, & -n_2 = -n_6 = -n_8 = r_3 = r_5 = r_6 = \beta - i\alpha \end{array}$$

Numerical values for the coefficients of the rational function for the approximate Riemann problem are obtained from equation (2.3) as

$$\mu = 1, \quad \alpha = 0.66952, \quad \beta = 1.6166, \quad S = -\overline{T} = \overline{M} = -N = 0.33463 - 0.03271\,i$$

The calculation of $\psi_+(x)$ is analogous.

4. We calculate the relative error in the approximate solution for $\psi_-(x)$. This function is considered to be an element of class $L_2(-\infty, 0)$ with a norm

$$\|\psi_-\|_{L_2}^2 = \int_{-\infty}^{0} |\psi_-(x)|^2\,dx$$

The function $\Psi^-(x)$, appearing as the Fourier transformation for $\psi_-(x)$, belongs to class $L_2(-\infty, \infty)$ and by virtue of the Parseval equality [3] we have

$$\|\psi_-\|_{L_2}^2 = \|\Psi^-\|_{L_2}^2 = \int_{-\infty}^{\infty} |\Psi^-(x)|^2\,dx$$

Let the known function $f_-(x)$ also belong to class $L_2(-\infty, 0)$ and the norms of $f_-(x)$ and $F^-(s)$ be determined analogously to the norms of $\psi_-(x)$ and $\Psi^-(x)$.

For calculation of the error we make use of the inequality

$$\delta = \frac{\|\varphi_- - \psi_-\|_{L_2}}{\|\psi_-\|_{L_2}} \leqslant \frac{\|K - K_*\|\,\|K_*^{-1}\|}{1 - \|K - K_*\|\,\|K_*^{-1}\|} \tag{4.1}$$

which comes from the general theory of approximate methods [6]. For this, $\varphi_-(x)$ is the solution of the exact equation $K\varphi_- = f_-$ corresponding (through a Fourier transformation) to the equality

$$\left[\frac{2\mu^2\Phi^-(x)}{\sqrt{x^2+\mu^2}\coth\sqrt{x^2+\mu^2} - \sqrt{x^2-\mu^2}\coth\sqrt{x^2-\mu^2}}\right]^- = -F^-(x) \tag{4.2}$$

and is obtained from the boundary condition (1.6) of the exact Riemann problem, while $\psi_-(x)$ is the solution of the approximate equation $K_*\psi_- = f_-$, corresponding to the equality

$$[2\mu^2\sqrt{x^2+\mu^2}\,P(x)\,\Psi^-(x)]^- = -F^-(x) \tag{4.3}$$

from the boundary condition (2.2) in the approximate Riemann problem.

The minus signs on the brackets in equations (4.2) and (4.3) have the same meaning as in equation (2.6).

We note also that K and K_* are linear operators (not required to be bounded but here bounded like the operator $K - K_*$) translating elements of the $L_2(-\infty, 0)$ space into elements of this same space. We note that the approximate equation $K_* \psi_- = f_-$ is easily solved and that the inverse (bounded) operator K_*^{-1} is known.

The errors in $\| K_*^{-1} \|$ and in $\| K - K_* \|$ are obtained as

$$\|K_*^{-1}\| \leqslant \frac{1}{2\mu^2} \max \left| \frac{X^-(x)}{\sqrt{x - i|\mu|}} \right| \max \frac{1}{|\sqrt{x + i|\mu|}\, X^+(x)|}$$

$$\|K - K_*\| \leqslant \max |2\mu^2 \sqrt{x^2 + \mu^2} [G_1(x) - P(x)]| \tag{4.4}$$

where the function $G_1(x)$ and $P(x)$ are taken from equalities (2.1) and (2.2), respectively.

It is seen from formulas (4.4) that the approximate coefficients of the Riemann problem must approach these values in order that the modulus of the difference between the exact and approximate coefficients be as small as possible.

By a substitution of the errors obtained for $\| K_*^{-1} \|$ and for $\|K - K_*\|$ into inequality (4.1) we find the relative error in the approximate solution for $\psi_-(x)$. For $\mu = 1$ and the values of a_0, a_1, b_0 and b_1 from (2.3) we obtain

$$\|K_*^{-1}\| \leqslant 0.336, \qquad \|K - K_*\| \leqslant 0.03, \qquad \delta \leqslant 0.0102$$

Consequently, in this case the relative error does not exceed 1.02 per cent.

5. We consider an example. In the boundary conditions (1.3) let

$$u_{yy}(x, 0) = f_-(x) = \begin{cases} 0 & (x > 0) \\ \sigma e^{\lambda x} & (x < 0) \end{cases} \qquad (\lambda > 0)$$

where σ is a real parameter. In this case we are led to a Riemann problem in the functions $\Phi^+(x)$ and $\Phi^-(x)$ with the boundary condition

$$\Phi^+(x) = \frac{-2\mu^2 \Phi^-(x)}{\sqrt{x^2 + \mu^2} \coth \sqrt{x^2 + \mu^2} - \sqrt{x^2 - \mu^2} \coth \sqrt{x^2 - \mu^2}} + \frac{i\sigma}{\sqrt{2\pi}(x - i\lambda)}$$

Upon setting $\mu = 1$ and substituting coefficients obtained for the function $P(x) = X^+(x)[X^-(x)]^{-1}$ we pass to the approximate Riemann problem

$$\Psi^+(x) = \frac{-2\sqrt{x^2 + 1}\, X^+(x)}{X^-(x)} \Psi^-(x) + \frac{i\sigma}{\sqrt{2\pi}(x - i\lambda)}$$

$$X^+(x) = \frac{(x+\alpha+i\beta)(x-\alpha+i\beta)}{(x+i\delta_1)(x+i\delta_2)}, \qquad X^-(x) = \frac{(x-i\delta_1)(x-i\delta_2)}{(x-\alpha-i\beta)(x+\alpha-i\beta)}$$

$$\alpha = 0.66952, \quad \beta = 1.6166, \quad \delta_1 = 1.5541, \quad \delta_2 = 1.6138$$

The solution of the approximate Riemann problem will be the functions

$$\Psi^+(x) = \frac{i\sigma}{\sqrt{2\pi}}\left[\frac{1}{x-i\lambda} - \frac{k\sqrt{(1+\lambda)\,i}\,\sqrt{x+i}\,(x+\alpha+i\beta)(x-\alpha+i\beta)}{(x-i\lambda)(x+i\delta_1)(x+i\delta_2)}\right]$$

$$\Psi^-(x) = \frac{-i\sigma k\sqrt{(1+\lambda)\,i}\,(x-i\delta_1)(x-i\delta_2)}{2\sqrt{2\pi}\sqrt{x-i}\,(x-i\lambda)(x-\alpha-i\beta)(x+\alpha-i\beta)}$$

$$\left(k = \frac{-i(\lambda+\delta_1)(\lambda+\delta_2)}{(1+\lambda)[\alpha^2+(\lambda+\beta)^2]}\right)$$

The inverse Fourier transformations give

$$\psi_+(x) = \frac{\sigma}{\sqrt{\pi}}\left\{\frac{c_1e^{-x}}{\sqrt{x}} + \sqrt{\pi}e^{\lambda x}\left[1-\operatorname{erf}\left(\sqrt{(1+\lambda)\,x}\right)\right] + c_2e^{-\delta_1 x}F\left(\sqrt{(\delta_1-1)\,x}\right) + \right.$$

$$\left. + c_3e^{-\delta_2 x}F\left(\sqrt{(\delta_2-1)\,x}\right)\right\}(x>0) \qquad \left(\operatorname{erf} z = \frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}\,dx, \quad F(z) = \int_0^z e^{x^2}\,dx\right)$$

$$\psi_-(x) = d_1\left\{\frac{d_2e^{\lambda x}F\left(\sqrt{(\lambda-1)\,|x|}\right)}{\sqrt{\lambda-1}} + \operatorname{Re}\left[\frac{d_3e^{(\beta-i\alpha)x}F\left(\sqrt{(\beta-1+i\alpha)\,|x|}\right)}{\sqrt{\beta-1+i\alpha}}\right]\right\}(x<0)$$

$$c_1 = \frac{-(\lambda+\delta_1)(\lambda+\delta_2)}{\sqrt{1+\lambda}\,[\alpha^2+(\beta+\lambda)^2]}, \qquad c_2 = \frac{2\sqrt{\delta_1-1}\,(\lambda+\delta_2)[\alpha^2+(\beta-\delta_1)^2]}{\sqrt{1+\lambda}\,(\delta_1-\delta_2)[\alpha^2+(\beta-\lambda)^2]}$$

$$c_3 = \frac{2\sqrt{\delta_2-1}\,(\lambda+\delta_1)[\alpha^2+(\beta-\delta_2)^2]}{\sqrt{1+\lambda}\,(-\delta_1+\delta_2)[\alpha^2+(\beta+\lambda)^2]}, \qquad d_1 = \frac{-\sigma(\lambda+\delta_1)(\lambda+\delta_2)}{\sqrt{\pi}\,(1+\lambda)[\alpha^2+(\beta+\lambda)^2]}$$

$$d_2 = \frac{(\lambda-\delta_1)(\lambda-\delta_2)}{\alpha^2+(\beta-\lambda)^2}, \qquad d_3 = \frac{\alpha^2-(\beta-\delta_1)(\beta-\delta_2)+i\alpha(2\beta-\delta_1-\delta_2)}{\alpha[\alpha+i(\beta-\lambda)]}$$

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